

Condensation in a Zero-Range Process with Fast Rates

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1 Introduction

Stochastic processes can be used to model random dynamic phenomena, where randomness is often introduced by a large number of neglected degrees of freedom, often too complicated to account for through measurement. A Zero-range process is a stochastic process, variations of which can be used to model traffic flow as done in [14]. In the Zero-range process, a constant number of particles move on a graph Λ . The particles move according to a jump rate, $g(\eta(x))$, where $\eta(x)$ is the number of particles at the site x . That means the probability of a jump from the site x only depends on how many particles are at x , and not on x specifically. For the finite lattice $\Lambda = \Lambda_L = \mathbb{Z}/L\mathbb{Z}$ and $X_L = \mathbb{N}^{\Lambda_L}$ consider

$$X_{L,N} = \{\eta \in X_L : \sum_{x \in \Lambda_L} \eta(x) = N\} \quad (1)$$

the compact set of all possible configurations $\eta \in \mathbb{N}^{\Lambda_L}$ of a Zero-range process at some time.

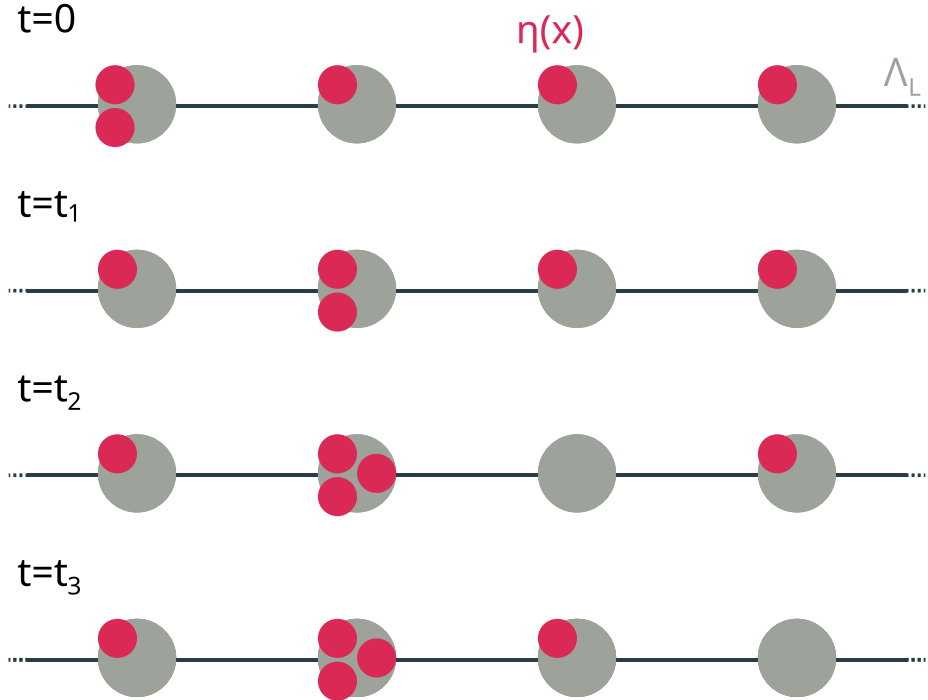


Figure 1: A possible realisation of a Zero-range process $(\eta_t)_{t \geq 0}$ at four times, Λ_L in grey, $\eta \in X_{L,N}$ in red.

To define the Zero-range process more rigorously and explore some of its properties, we will first briefly introduce Markov processes, Semigroups, and Generators which are studied in more detail in [12]. Finally, we will examine some

properties of a Zero-range process numerically; the variation we will be examining has a high jump rate for exactly two particles at a site and a constant low rate for any other number of particles.

We will find that there exists a unique stationary distribution $\pi_{L,N}$ of the process on $X_{L,N}$ that converges to a product measure in the thermodynamic limit, so for $L, N \rightarrow \infty$ if $N/L \rightarrow \rho$, where ρ is the density at a site of the lattice; We will see that the largest density of the limiting product measure is $1/2$, and if $\rho > 1/2$ the system exhibits condensation.

This is known as the equivalence of ensembles, meaning, for large L we can approximate $\pi_{L,N}$ by ν_ϕ , which has a simpler product structure as seen in section 4. A more in-depth discussion of stationary measures and equivalence of ensembles can be found in [10, 6, 5, 4]

In the case of condensation for $\rho > 1/2$ we observe, that few sites have many particles and the majority has just a few, in this case, one or less. We call sites with many particles clusters and all cluster sites together the condensation. If a site has only a few particles, its particles are part of the bulk. In the context of this thesis, bulk sites have at most one particle. Condensation, generally, has been studied extensively in works such as [9, 7, 4] and, specifically in the context of Zero-range processes in [1, 2, 11, 13, 6, 5].

For our numerical findings in the case of ρ greater $1/2$ multiple particles will be present at some sites, called clusters. The expected number of particles on cluster sites is proportional to $\sqrt{L}$. Approximately half of the sites in the bulk have one particle for ρ greater $1/2$. For ρ smaller or equal $1/2$ we observe different behavior; there are no clusters and the sites with exactly one particle will occupy N sites on the lattice.

2 Markov Processes

A continuous time stochastic process $(\eta_t)_{t \geq 0}$ is a random configuration at any time t , with a configuration being any element in the state space $X = S^\Lambda$. The state space X is the set of functions from a vertex set Λ of a strongly connected graph to a local state space S . $\eta_t(x)$ is the state of site x at time t .

Λ should be strongly connected to avoid degeneracies, like two mutually isolated parts of the graph. Usually, the edges of the graph are not explicitly specified.

In the case of the Zero-range process (ZRP) the local state space is $\mathbb{N}$. Note that $X = \mathbb{N}^\Lambda$ is generally not a compact set. However, $X_{L,N}$ from (1) is. For the following theory, X will be a compact metric space.

We can characterize the random variables $(\eta_t)_{t \geq 0}$ by a probability measure $\mathbb{P}$ on

$$D[0, \infty) := \{f : [0, \infty) \rightarrow X, f \text{ right continuous with left limits}\}.$$

$D[0, \infty)$ is called the canonical path space. We need a σ -algebra $\mathcal{F}$ on $D[0, \infty)$, which will be the smallest σ -algebra on $D[0, \infty)$ such that $\eta \mapsto \eta_s$ for all $s \geq 0$ is measurable and a filtration $\mathcal{F}_t$ of $\mathcal{F}$ such that $\eta \mapsto \eta_s$ for $0 \leq s \leq t$ is measurable. With this setup, we can define Markov processes.

Definition (Markov process)

A (homogenous) Markov process $(\eta_t : t \geq 0)$ on X is characterized by a collection $(\mathbb{P}^\zeta : \zeta \in X)$ of probability measures on $D[0, \infty)$ with the following properties:

1. $\mathbb{P}^\zeta(\eta_0 = \zeta) = 1$ for all $\zeta \in X$
2. $\mathbb{P}^\zeta(\eta_{t+} \in A | \mathcal{F}_t) = \mathbb{P}^{\eta_t}(A)$ for all $\eta \in X, A \in \mathcal{F}$ and $t > 0$ (Markov property)
3. The mapping $\eta \mapsto \mathbb{P}^\eta(A)$ is measurable for every $A \in \mathcal{F}$

If X additionally is countable, we will call a Markov process on X a Markov chain.

3 Semigroups and Generators

Some definitions, propositions, and theorems in this section are taken from [12]. In this section, we will define the generator of a process and use generators to define stationary measures of a process; we will find that Feller processes are equivalent to their generator. We can later use these definitions to get more detail on the long time behavior of a ZRP.

Now let X be a compact metric space and let $C(X) = \{f : X \rightarrow \mathbb{R} \text{ continuous}\}$, notice $(C(X), \|\cdot\|_\infty)$ is a Banach space.

Definition

For a given Markov process $(\eta_t : t \geq 0)$ on X , for each $t \geq 0$ we define the operator

$$S(t) : C(X) \rightarrow C(X) \text{ by } (S(t)f)(\zeta) := \mathbb{E}^\zeta f(\eta_t).$$

If $S(t)f \in C(X)$, for all $f \in C(X)$ the process is called Feller process.

Notice that $C(X)$ is dual to $\mathcal{M}_1(X) = \{m | m \text{ is a probability measure on } X\}$ which means that we can characterise the distribution of η_t by $\mathbb{E}^\zeta(f(\eta_t))$ for all $f \in C(X)$.¹

The operator $S(t)$ does exactly that, so we now have an object that is equivalent to our process, which we will see within the next theorem, and based on that, we are later able to define generators.

Proposition (Markov semigroup)

Let $(\eta_t : t \geq 0)$ be a Feller process on X . Then the family $(S(t) : t \geq 0)$ is a Markov semigroup, i.e.

1. $S(0) = Id$
2. $t \mapsto S(t)f$ is right continuous for all $f \in C(X)$
3. $S(t+s)f = S(t)S(s)f$ for all $f \in C(X), s, t \geq 0$
4. $S(t)1 = 1$ for all $t \geq 0$
5. $S(t)f \geq 0$ for all non-negative $f \in C(X)$

Proof: see [12] Proposition 1.3

Theorem

Suppose $(S(t) : t \geq 0)$ is a Markov semigroup on $C(X)$. Then there exists a unique (Feller) Markov process $(\eta_t : t \geq 0)$ on X such that

$$\mathbb{E}^\zeta f(\eta_t) = S(t)f(\zeta) \text{ for all } f \in C(X), \zeta \in X \text{ and } t \geq 0.$$

Proof: see [12] Theorem 1.4

Definition (Generator)

The generator $\mathcal{L} : D_{\mathcal{L}} \rightarrow C(X)$ for the process $(S(t) : t \geq 0)$ is given by

$$\mathcal{L}f := \lim_{t \searrow 0} \frac{S(t)f - f}{t} \text{ for } f \in D_{\mathcal{L}} \text{ the limit w.r.t } \|\cdot\|_\infty,$$

where the domain $D_{\mathcal{L}} \subseteq C(X)$ is the set of functions for which the limit exists.

¹see Riesz representation theorem in [17] Theorem 2.14

Proposition (Markov generator)

$\mathcal{L}$ as defined above is a Markov generator, i.e.

1. $1 \in D_{\mathcal{L}}$ and $\mathcal{L}1 = 0$
2. for $f \in D_{\mathcal{L}}, \lambda \geq 0 : \min_{\zeta \in X} f(\zeta) \geq \min_{\zeta \in X} (f - \lambda \mathcal{L}f)(\zeta)$
3. $D_{\mathcal{L}}$ is dense in $C(X)$ and the range $\mathcal{R}(Id - \lambda \mathcal{L}) = C(X)$ for sufficiently small $\lambda > 0$

Proof: see [12] Proposition 1.5

Theorem (Hille-Yosida)

There is a one-to-one correspondence between Markov generators and semigroups on $C(X)$, given by

$$\mathcal{L}f := \lim_{t \searrow 0} \frac{S(t)f - f}{t} \text{ for } f \in D_{\mathcal{L}}$$

and

$$S(t)f := \lim_{n \rightarrow \infty} \left(Id - \frac{t}{n} \mathcal{L} \right)^{-n} f \text{ for } f \in C(X), t \geq 0.$$

Proof: see [15] Theorem I.2.9

We now have defined a generator and found that it is, in fact, equivalent to the operator $S(t)$, which is the groundwork for the following result of what it means for a process to be stationary in terms of its generator. We will use this in section 4 to find the stationary distribution of the Zero-range process.

Definition (Stationary probability measure)

A normalised probability measure μ on X is stationary (or invariant) if

$$\int_X S(t)f d\mu = \int_X f d\mu \text{ or shorter } \mu(S(t)f) = \mu(f) \text{ for all } f \in C(X).$$

And the set of all stationary measures of a process is denoted by $\mathcal{I}$.

Definition (Cylinder Test Function)

For $X = S^\Lambda$ with $S \subset \mathbb{N}$, $f \in C(X)$ is called cylinder test function if there exists a finite $\Delta \subset \Lambda$ such that for any $\eta, \zeta \in X$ $f(\eta) = f(\zeta)$ if $\eta|_\Delta = \zeta|_\Delta$.
We write $C_0(X) \subset C(X)$ for the set of all cylinder test functions.

Proposition

Consider a Feller process on a compact state space X with generator $\mathcal{L}$.
Then

$$\mu \in \mathcal{I} \iff \mu(\mathcal{L}f) = 0 \text{ for all } f \in C_0(X).$$

Proof: see [12] Proposition 1.8

4 Zero-Range Processes

Definition (Zero-Range Process)

Let $X = \mathbb{N}^\Lambda$ with finite lattice Λ be the state space and $p(x, y)$ irreducible transition probabilities of a single random walker on Λ with $p(x, x) = 0$, called jump probabilities. For each $x \in \Lambda$ define the jump rates $g_x : \mathbb{N} \rightarrow [0, \infty)$ as a non-negative function of the number of particles $\eta(x)$ at site x , where

$$\forall x \in \Lambda : g_x(n) = 0 \iff n = 0.$$

Then the process $(\eta_t : t \geq 0)$ on X defined by the generator

$$\mathcal{L}f(\eta) = \sum_{x, y \in \Lambda} g_x(\eta(x))p(x, y)(f(\eta^{x \rightarrow y}) - f(\eta))$$

where

$$\eta^{x \rightarrow y}(z) = \begin{cases} \eta(x) - 1 & \text{for } z = x \\ \eta(y) + 1 & \text{for } z = y \\ \eta(z) & \text{otherwise} \end{cases}$$

is called a zero-range process.

$\eta^{x \rightarrow y}$ is the configuration after a particle jumps from site x to site y with configuration η before the jump.

Since we introduced generators on compact sets, some modifications to the definition of generators would have to be made if we were interested in infinite lattices.¹ However, we are only interested in the finite Lattice $\Lambda_L = \mathbb{Z}/L\mathbb{Z}$ and $X_L = \mathbb{N}^{\Lambda_L}$, this is still not compact but can be treated essentially the same with small modifications, which are discussed in [12] p. 46. We also limit ourselves to symmetrical transition probabilities $p(x, y) = p(y, x)$ and site-independent jump rates $g_x(n) = g(n)$.

Firstly, we will investigate the stationary measure for this generator on X_L .

4.1 The Grand-Canonical Ensemble

Definition (grand-canonical ensemble)

We call $w(n)$ the weights and $z(\phi)$ the local normalization, or partition function

$$w(n) = \prod_{k=1}^n \frac{1}{g(k)}$$

$$z(\phi) = \sum_{n=0}^{\infty} w(n) \phi^n,$$

with ϕ_c the radius of convergence of $z(\phi)$. Also, let $d\eta$ be the counting measure on X_L .

Then for $\phi \geq 0$, the let $\nu_{L,\phi}$ be a product measure on $X_L = \mathbb{N}^{\Lambda_L}$

$$\nu_{L,\phi}(d\eta) = \prod_{x \in \Lambda_L} \nu_{\phi}^x(\eta(x)) d\eta = \frac{1}{z(\phi)^L} \prod_{x \in \Lambda_L} w(\eta(x)) \phi^{\eta(x)} \quad (2)$$

with marginals

$$\nu_{L,\phi}(\eta(x) = n) = \nu_{\phi}^x(n) = \frac{w(n) \phi^n}{z(\phi)}.$$

We call $\{\nu_{L,\phi} : \phi \in [0, \phi_c)\}$ grand-canonical ensemble.

Theorem

$\nu_{L,\phi}$ is stationary w.r.t. $\mathcal{L}f(\eta)$, the generator of the ZRP, provided that the local normalization $z(\phi) < \infty$. So $\phi \in [0, \phi_c)$.

Proof:

For this proof let $\nu_{\phi} := \nu_{L,\phi}$, for simplicity.

¹see [12] Theorem 3.1

Notice $w(n+1) = \frac{w(n)}{g(n+1)}$ and thus we have:

$$\begin{aligned}\nu_\phi^x(n+1) &= \frac{\phi^{n+1}w(n+1)}{z(\phi)} \\ &= \frac{\phi}{g(n+1)} \frac{\phi^n w(n)}{z(\phi)} \\ &= \frac{\phi}{g(n+1)} \nu_\phi^x(n)\end{aligned}$$

from this we can derive:

$$\begin{aligned}\nu_\phi(\eta^{y \rightarrow x}) &= \nu_\phi^x(\eta(x)+1) \nu_\phi^y(\eta(y)-1) \prod_{z \neq x, y} \nu_\phi^z(\eta(z)) \\ &= \frac{g(\eta(y))}{g(\eta(x)+1)} \nu_\phi^x(\eta(x)) \nu_\phi^y(\eta(y)) \prod_{z \neq x, y} \nu_\phi^z(\eta(z)) \\ &= \frac{g(\eta(y))}{g(\eta(x)+1)} \nu_\phi(\eta).\end{aligned}$$

So

$$\nu_\phi(\eta^{y \rightarrow x}) g(\eta(x)+1) = \nu_\phi(\eta) g(\eta(y)). \quad (3)$$

We also notice that:

$$\sum_{\eta \in X} g(\eta(x)) p(x, y) f(\eta^{x \rightarrow y}) \nu_\phi(\eta) \quad (4)$$

$$= \sum_{\eta \in X} g(\eta^{y \rightarrow x}(x)) p(x, y) f((\eta^{y \rightarrow x})^{x \rightarrow y}) \nu_\phi(\eta^{y \rightarrow x}) \quad (5)$$

$$= \sum_{\eta \in X} g(\eta(x)+1) p(x, y) f(\eta) \nu_\phi(\eta^{y \rightarrow x}). \quad (6)$$

The first equality follows from the fact that $\eta \mapsto \eta^{y \rightarrow x}$ is a bijection on X with the inverse being $\eta \mapsto \eta^{x \rightarrow y}$.

To show that ν_ϕ is stationary we have to show that $\nu_\phi(\mathcal{L}f) = 0$. So

$$\begin{aligned}
& \nu_\phi(\mathcal{L}f) \\
&= \sum_{\eta \in X} \sum_{x, y \in \Lambda_L} g(\eta(x)) p(x, y) (f(\eta^{x \rightarrow y}) - f(\eta)) \nu_\phi(\eta) \\
&= \sum_{x, y \in \Lambda_L} \sum_{\eta \in X} g(\eta(x)) p(x, y) (f(\eta^{x \rightarrow y}) - f(\eta)) \nu_\phi(\eta) \\
&= \sum_{x, y \in \Lambda_L} \left[\left(\sum_{\eta \in X} g(\eta(x)) p(x, y) f(\eta^{x \rightarrow y}) \nu_\phi(\eta) \right) \right. \\
&\quad \left. - \left(\sum_{\eta \in X} g(\eta(x)) p(x, y) f(\eta) \nu_\phi(\eta) \right) \right] \\
&\stackrel{(6)}{=} \sum_{x, y \in \Lambda_L} \left[\left(\sum_{\eta \in X} g(\eta(x) + 1) p(x, y) f(\eta) \nu_\phi(\eta^{y \rightarrow x}) \right) \right. \\
&\quad \left. - \left(\sum_{\eta \in X} g(\eta(x)) p(x, y) f(\eta) \nu_\phi(\eta) \right) \right] \\
&= \sum_{x, y \in \Lambda_L} \sum_{\eta \in X} p(x, y) f(\eta) \left(g(\eta(x) + 1) \nu_\phi(\eta^{y \rightarrow x}) - g(\eta(x)) \nu_\phi(\eta) \right) \\
&\stackrel{(3)}{=} \sum_{x, y \in \Lambda_L} \sum_{\eta \in X} p(x, y) f(\eta) \nu_\phi(\eta) \left(g(\eta(y)) - g(\eta(x)) \right) \\
&= \sum_{\eta \in X} f(\eta) \nu_\phi(\eta) \sum_{x, y \in \Lambda_L} p(x, y) \left(g(\eta(y)) - g(\eta(x)) \right).
\end{aligned}$$

Now since $p(y, x) = p(x, y)$, $p(x, y) (g(\eta(y)) - g(\eta(x)))$ is an odd function in x and y and we get the desired result.

q.e.d.

Secondly, we will show the relationship between the partition function $z(\phi)$ and the density $R(\phi) = \mathbb{E}_{\nu_\phi^\pi}(\eta(x))$. As discussed in the introduction, the invertibility of the density is important for the convergence result $\pi_{L,N} \rightarrow \nu_\phi$; we will go into more detail about it as soon as we formally introduce $\pi_{L,N}$, the measure on $X_{L,N}$ and ν_ϕ the measure on the whole lattice $\mathbb{N}^{\mathbb{N}}$.

Proposition

Let $R(\phi) := \mathbb{E}_{\nu_\phi^x}(\eta(x))$ be the density at any site and ϕ_c the radius of convergence of $z(\phi)$. Then for $\phi \in [0, \phi_c)$ the following is true

$$R(\phi) = \phi \frac{d}{d\phi} \ln(z(\phi)). \quad (7)$$

Proof:

Remember

$$z(\phi) = \sum_{k=0}^{\infty} w(k) \phi^k.$$

Also notice that multiplying the coefficients $w(k)$ by k will not change the radius of convergence, so $R(\phi)$ has the same radius of convergence as $z(\phi)$.

$$\begin{aligned} R(\phi) &= \mathbb{E}_{\nu_\phi^x}(\eta(x)) = \sum_{k=0}^{\infty} k \nu_\phi^x(\eta(x) = k) = \sum_{k=1}^{\infty} k \nu_\phi^x(\eta(x) = k) \\ &= \sum_{k=1}^{\infty} k \frac{w(k) \phi^k}{z(\phi)} = \frac{\phi}{z(\phi)} \sum_{k=1}^{\infty} k \phi^{k-1} w(k) \\ &= \frac{\phi}{z(\phi)} \sum_{k=1}^{\infty} \frac{d}{d\phi} (\phi^k) w(k) = \frac{\phi}{z(\phi)} \sum_{k=1}^{\infty} \frac{d}{d\phi} (\phi^k w(k)) \\ &\stackrel{(*)}{=} \frac{\phi}{z(\phi)} \frac{d}{d\phi} \sum_{k=0}^{\infty} \phi^k w(k) = \frac{\phi}{z(\phi)} \frac{d}{d\phi} z(\phi) \\ &= \phi \frac{d}{d\phi} \ln(z(\phi)) \end{aligned}$$

Since as we know we can differentiate power series part-wise, to get their derivative, as used at (*).

q.e.d.

4.2 The Canonical Ensemble

Now since the lattice we consider is finite, $p(x, y)$ is irreducible, and $g(n) = 0$ only if $n = 0$, the ZRPs we consider are irreducible finite state Markov chains on

$$X_{L,N} = \{\eta \in \mathbb{N}^{\Lambda_L} : \sum_{x \in \Lambda_L} \eta(x) = N\}$$

and thus have a unique stationary distribution $\pi_{L,N}$ on $X_{L,N}$.¹ Since $\nu_{L,\phi}(d\eta|X_{L,N})$ is stationary on $X_{L,N}$ and the stationary distribution $\pi_{L,N}$ is unique on $X_{L,N}$ they must be equal:

$$\begin{aligned}\pi_{L,N}(d\eta) &= \nu_{L,\phi}(d\eta|X_{L,N}) \\ &= \mathbb{1}_{X_{L,N}}(\eta) \frac{\nu_{L,\phi}(d\eta)}{\nu_{L,\phi}(X_{L,N})}\end{aligned}$$

Proposition

The unique stationary distribution $\pi_{L,N}$ of the ZRP on $X_{L,N}$ is:

$$\pi_{L,N}(d\eta) = \frac{\mathbb{1}_{X_{L,N}}(\eta)}{Z_{L,N}} \prod_{x \in \Lambda_L} w(\eta(x)) d\eta, \quad (8)$$

where $Z_{L,N} = \sum_{\tilde{\eta} \in X_{L,N}} \prod_x w(\tilde{\eta}(x))$.

We call $\{\pi_{L,N} : N \in \mathbb{N}\}$ canonical ensemble.

Proof: also see [12] p.50 (3.39)

For simplicity of notation, we will use the short-hands

$$\begin{aligned}\sum \eta &:= \sum_{x \in \Lambda_L} \eta(x) \\ \prod w\eta &:= \prod_{x \in \Lambda_L} w(\eta(x))\end{aligned}$$

and calculate the quotients separately:

$$\begin{aligned}\mathbb{1}_{X_{N,L}}(\eta) \nu_{L,\phi}(d\eta) &= \mathbb{1}_{X_{N,L}}(\eta) \prod_{x \in \Lambda_L} \frac{w(\eta(x)) \phi^{\eta(x)}}{z(\phi)} d\eta \\ &= \mathbb{1}_{X_{N,L}}(\eta) \frac{\phi^{\sum \eta} \prod w\eta}{z(\phi)^L} d\eta \\ &\stackrel{(*)}{=} \mathbb{1}_{X_{N,L}}(\eta) \frac{\phi^N \prod w\eta}{z(\phi)^L} d\eta\end{aligned}$$

¹see [16] Theorem 3.5.2.

If $\eta \in X_{N,L}$ we know that $\sum_{x \in \Lambda_L} \eta(x) = N$, which is what we used at (*).

$$\begin{aligned} \nu_{L,\phi}(X_{L,N}) &= \sum_{\tilde{\eta} \in X_{L,N}} \prod_{x \in \Lambda_L} \frac{w(\tilde{\eta}(x)) \phi^{\tilde{\eta}(x)}}{z(\phi)} \\ &= \sum_{\tilde{\eta} \in X_{L,N}} \frac{\phi^N \prod w \tilde{\eta}}{z(\phi)^L} \\ &= \frac{\phi^N \sum_{\tilde{\eta} \in X_{L,N}} \prod w \tilde{\eta}}{z(\phi)^L} \end{aligned}$$

Now plugging all of that in leaves us with the following

$$\begin{aligned} \pi_{L,N}(d\eta) &= \mathbb{1}_{X_{L,N}}(\eta) \frac{\phi^N \prod w \eta}{z(\phi)^L} \frac{z(\phi)^L}{\phi^N \sum_{\tilde{\eta} \in X_{L,N}} \prod w \tilde{\eta}} d\eta \\ &= \frac{\mathbb{1}_{X_{L,N}}(\eta)}{Z_{L,N}} \prod_{x \in \Lambda_L} w(\eta(x)) d\eta, \end{aligned}$$

where $Z_{L,N} = \sum_{\tilde{\eta} \in X_{L,N}} \prod w \tilde{\eta}$.

q.e.d.

Proposition ($\pi_{L,N}(\eta(x))$)

The expected number of particles at a site x for the canonical measures (8) is given by

$$\pi_{L,N}(\eta(x)) = \frac{N}{L}.$$

Proof: also see [3] p. 2

Because of spacial homogeneity:

$$\pi_{L,N}(\eta(x)) = \frac{1}{L} \sum_{x \in \Lambda_L} \pi_{L,N}(\eta(x))$$

and since $\sum_{x \in \Lambda_L} \eta(x) = N$, we get:

$$\pi_{L,N}(\eta(x)) = \frac{1}{L} \pi_{L,N} \left(\sum_{x \in \Lambda_L} \eta(x) \right) = \frac{N}{L}$$

q.e.d.

4.3 Equivalence of Ensembles

Proposition

Consider the family $(\nu_\phi^x : x \in \mathbb{N})$ of probability measures on $(\mathbb{R}, \mathcal{B})$, where $\mathcal{B}$ is the borel σ -algebra on $\mathbb{R}$. There exists a unique probability measure ν_ϕ on $(\mathbb{R}^\mathbb{N}, \mathcal{B}^{\otimes \mathbb{N}})$, $\mathcal{B}^{\otimes \mathbb{N}}$ being the product σ -algebra, for wich the projections $Y_i : \mathbb{R}^\mathbb{N} \rightarrow \mathbb{R}$ are independent with marginals ν_ϕ^x :

$$\nu_\phi(Y_x \in A_x \text{ for all } x \in J) = \prod_{x \in J} \nu_\phi^x(A_x)$$

for all finite, non-empty $J \subset \mathbb{N}$.

Proof: see [8] Corollary 3.27

The following proposition is almost directly taken from [3]. Before this theorem, we need to consider that the weights can, and later will be, dependent on the lattice size L . Considering the behavior of w dependent on L , therefore, comes naturally. So far we only considered results for fixed lattice size L , but for the next result, we need some properties that depend on the behavior of the weights as the lattice size is increasing.

Theorem (Equivalence of Ensembles in the Thermodynamic Limit)

Consider weights, w , with $w(n) \rightarrow \tilde{w}(n)$ for $L \rightarrow \infty$ and the following properties:

- $\forall n \geq 0 : w(n) > 0$
- $\sum_{n=0}^{\infty} \tilde{w}(n) = 1$ and either $\tilde{w}(0) = 1$ or
 $\tilde{w}(0) > 0$ and $\sup_n [\tilde{w}(n-1) \wedge \tilde{w}(n)] > 0$,
- $\forall a > 0 : \lim_{L \rightarrow \infty} \frac{\ln(w(aL))}{L} = 0$ and $\sum_{n=0}^{\infty} n^2 \tilde{w}(n) < \infty$,

then, for each $M \subset \mathbb{N}_0$, with $\#M = m < \infty$, we have

$$\pi_{L,N}^M \xrightarrow{d} \begin{cases} \nu_{\phi}^{\otimes m} & \text{if } \rho < \rho_c \text{ with } \phi : \lim_{L \rightarrow \infty} R(\phi) = \rho \\ \nu_{\phi_c}^{\otimes m} & \text{if } \rho \geq \rho_c \end{cases}$$

in the thermodynamic limit $L, N \rightarrow \infty, N/L \rightarrow \rho$ where $\pi_{L,N}^M$ is the marginal of $\pi_{L,N}$ on M , and $\rho_c = \lim_{\phi \uparrow \phi_c} \lim_{L \rightarrow \infty} R(\phi)$.

Proof: see [3] Proposition A.1

Remarks:

- Notice that we can scale w first, to meet the restrictions of the theorem. We will later specify w to be $w(n) = \mathbb{1}_{\{0,1\}}(n) + L^{-1} \mathbb{1}_{\{2,\dots\}}(n)$ which converges to $\tilde{w}(n) = \mathbb{1}_{\{0,1\}}(n)$ for $L \rightarrow \infty$. This w does not fulfill the limitations, however, if we scale the weights $w'(n) := w(n)/2 \rightarrow \tilde{w}'(n)$, we find that the process is still functionally the same but the weights meet the restrictions:

$$\begin{aligned} & \forall n \geq 0 : w'(n) > 1/2L > 0 \checkmark \\ & \sum_{n=0}^{\infty} \tilde{w}'(n) = 1/2 + 1/2 + 0 + \dots = 1 \\ & \tilde{w}'(0) > 0 \text{ and } \sup_n [\tilde{w}'(n-1) \wedge \tilde{w}'(n)] = \tilde{w}'(0) \wedge \tilde{w}'(1) = \frac{1}{2} > 0 \checkmark \\ & \forall a > 0 : \lim_{L \rightarrow \infty} \frac{\ln(w'(aL))}{L} = \lim_{L \rightarrow \infty} \frac{-\ln(2L)}{L} \stackrel{\text{IH}}{=} \lim_{L \rightarrow \infty} \frac{-1}{2L} = 0 \\ & \sum_{n=0}^{\infty} n^2 \tilde{w}'(n) = 1 < \infty \checkmark \end{aligned}$$

Also see [3] Remark 1.3 for a more detailed discussion.

- With this theorem, we can see that the inverse function of ρ is quite important in determining the behavior of the process. For large enough L and N we can approximate $\pi_{L,N}$ through ν_ϕ . As we see in figure 2 we can only solve $R(\phi) = \rho$ for $\rho \leq \rho_c$ for $L \rightarrow \infty$.

When we look at the numerical results in the following section, we will find that whenever $\pi_{L,N} \approx \nu_\phi$ – for large L and N with $\rho \leq \rho_c$ – the particles have enough space to take individual spots on the lattice when that is not the case we will find particles clustering.

Now in section 5 we will be looking at a more specific setup, the results for the grand-canonical and canonical ensemble are independent of this (see (2) and (8)). With transition probabilities such that from any point on the lattice particles are able to jump to any other with equal probability

$$p(x, y) = \begin{cases} \frac{1}{L-1} & x \neq y \\ 0 & x = y \end{cases}$$

and the jump rates g , as described in the introduction

$$g(n) = \begin{cases} 0 & \text{for } n = 0 \\ L & \text{for } n = 2 \\ 1 & \text{otherwise} \end{cases}.$$

From this, the explicit formula for the weight function and partition function follow immediately:

$$\begin{aligned} w(n) &= \prod_{k=1}^n \frac{1}{g(k)} = \begin{cases} 1 & \text{for } n \leq 1 \\ \frac{1}{L} & \text{otherwise} \end{cases} \rightarrow \begin{cases} 1 & \text{for } n \leq 1 \\ 0 & \text{otherwise} \end{cases} \text{ for } L \rightarrow \infty \\ z(\phi) &= \sum_{n=0}^{\infty} w(n) \phi^n = (1 + \phi) \left(1 - \frac{1}{L}\right) + \frac{1}{L(1 - \phi)} \\ &\rightarrow 1 + \phi \text{ for } L \rightarrow \infty \end{aligned}$$

For the marginals ν_ϕ^x of ν_ϕ we find for $L \rightarrow \infty$

$$\nu_\phi^x = \frac{\phi^n w(n)}{z(\phi)} \rightarrow \begin{cases} \frac{1}{1+\phi} & \text{for } n = 0 \\ \frac{\phi}{1+\phi} & \text{for } n = 1 \\ 0 & \text{otherwise} \end{cases}.$$

We can also explicitly calculate the density

$$\begin{aligned} R(\phi) &= \phi \frac{d}{d\phi} \ln(z(\phi)) \\ &\rightarrow \frac{\phi}{1 + \phi} \text{ for } L \rightarrow \infty \text{ and } \phi < \phi_c = 1, \end{aligned}$$

we examined in 7. $\frac{\phi}{1+\phi}$ exists for all $\phi \geq 0$, but we do not have convergence for $\phi \geq \phi_c$. We can now define the inverse of $\lim_{L \rightarrow \infty} R(\cdot)$, as

$$\Phi(\rho) = \frac{\rho}{1 - \rho}.$$

We know that the weights that resulted from our choice of jump rates satisfy the conditions of the theorem *Equivalence of Ensembles in the Thermodynamic Limit*, so we can now use this theorem to make some predictions about the behavior of the process in the next section.

In the thermodynamic limit $L, N \rightarrow \infty, N/L \rightarrow \rho < \rho_c$ we get:

$$\begin{aligned} \pi_{L,N}^{\{x\}}(\eta(x) = 0) &\rightarrow \nu_{\Phi(\rho)}^x(\eta(x) = 0) = \frac{1}{1 + \Phi(\rho)} \\ \pi_{L,N}^{\{x\}}(\eta(x) = 1) &\rightarrow \nu_{\Phi(\rho)}^x(\eta(x) = 1) = \frac{\Phi(\rho)}{1 + \Phi(\rho)} = \rho \end{aligned}$$

and for $\rho \geq \rho_c$:

$$\begin{aligned} \pi_{L,N}^{\{x\}}(\eta(x) = 0) &\rightarrow \nu_{\phi_c}^x(\eta(x) = 0) = \frac{1}{2} \\ \pi_{L,N}^{\{x\}}(\eta(x) = 1) &\rightarrow \nu_{\phi_c}^x(\eta(x) = 1) = \frac{1}{2} \end{aligned}$$

and in either case for $n > 1$:

$$\pi_{L,N}^{\{x\}}(\eta(x) = n) \rightarrow 0$$

We can also look at the density in the case of $\rho \geq \rho_c$ by considering the implications of the above, in the thermodynamic limit $\pi_{L,N}^{\{x\}}(\eta(x) = n)$ is only greater than 0 for $n \leq 1$ so the density, in that case, is $\frac{1}{2}$, this will later be reflected numerically in section 5.1. This also means that in the case of $\rho \geq \rho_c$ the excess particles must be condensed in a vanishing volume fraction of sites, this will later also be mirrored in the findings of sections 5.2.3 and 6.

The formulas we get for $R(\phi)$ in the case of $L < \infty$ are not particularly enlightening, so we will just plot some of them in figure 2.

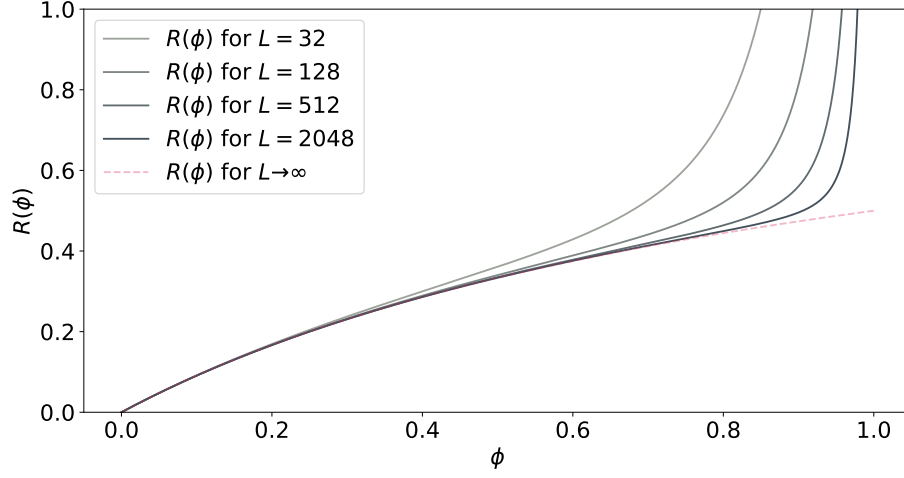


Figure 2: $R(\phi)$ for different values of L

We can see that the convergence radius for $R(\phi)$ is $\phi_c = 1$ and for $L \rightarrow \infty$ we will be able to invert $R(\phi)$ up until $\rho_c = \frac{1}{2}$.

5 Numerical Results

We will now look at realizations of $\eta \sim \pi_{L,N}$. To get a impression of what a possible η might look like, we will plot

$$H_\eta(k) = \sum_{x=1}^k \eta(x) \text{ for } k = 1 \dots L$$

with $L = N = 2048$ in figure 3.

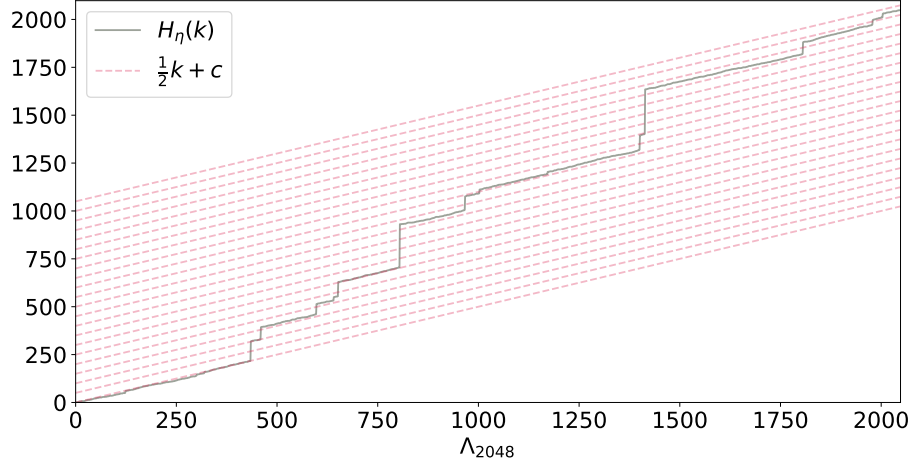


Figure 3: $H_\eta(k) = \sum_{x=1}^k \eta(x)$ for $k = 1 \dots 2048, L = N = 2048$

Notice the jumps in the graph; these are called clusters. Everywhere else, the graph seems to grow with a slope of $\frac{1}{2}$ so at sites with at most one particle, we are equally likely to find either a single particle or none at all: $\frac{\#\{x \in \Lambda_L : \eta(x)=1\}}{\#\{x \in \Lambda_L : \eta(x) \leq 1\}} \approx \frac{1}{2}$, that is the bulk density.

Before we can look at those attributes in more detail we will define the sample average and sample standard deviation.

Definition (sample average and standard deviation)

For any finite $X \subset \mathbb{R}$, called the sample,

$$M(X) := \frac{1}{\#X} \sum_{x \in X} x$$

is called sample average,

$$S(X) := \sqrt{\frac{1}{\#X - 1} \sum_{x \in X} (x - M(X))^2}$$

is called sample standard deviation.

For any respectively finite $X_1, \dots, X_n \subset \mathbb{R}$, ($n \in \mathbb{N}$):

$$M(\{X_1, \dots, X_n\}) := M\left(\bigcup_{i=1}^n X_i\right)$$

$$S(\{X_1, \dots, X_n\}) := S\left(\bigcup_{i=1}^n X_i\right)$$

5.1 Bulk Density

For $\eta \in X_{L,N}$

$$b_\eta = \frac{\#\{x \in \Lambda_L : \eta(x) = 1\}}{\#\{x \in \Lambda_L : \eta(x) \leq 1\}} \quad (9)$$

is called the bulk density,

$$\tilde{b}_\eta = \frac{\#\{x \in \Lambda_L : \eta(x) \leq 1\}}{L} \quad (10)$$

is the fraction of sites without clusters.

For $n = 50$ realizations η_i and $N = L = 1024$ i.e. $\rho = 1 > \frac{1}{2}$ we can calculate sample average and sample standard deviation:

$$M((b_{\eta_i})_{i=1}^n) = 0.49045373083125354$$

$$S((b_{\eta_i})_{i=1}^n) = 0.014701574797048986$$

It seems the average bulk density is $\frac{1}{2}$, exactly as we observed in figure 3 and expected from the theorem on equivalence of ensembles in section 4.3. In earlier

sections, we discussed the role of $\rho := \frac{N}{L}$ and that we would expect different behavior from realizations depending on whether $\rho \leq 1/2$ or not.

To test this, we can realize η with varying N and L . If we do this multiple times for each N and L combination, we get an average and standard deviation for each value of ρ as seen in figure 4.

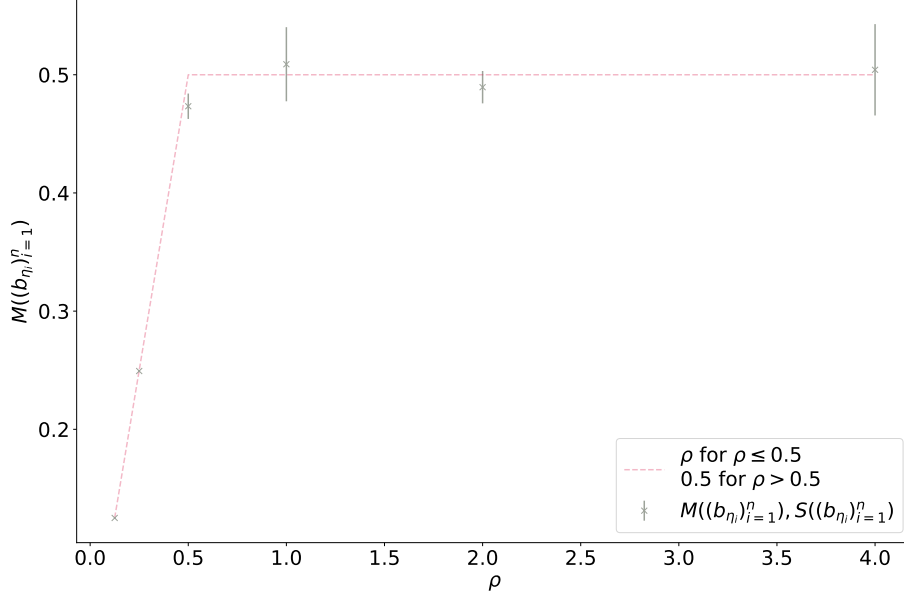


Figure 4: The average bulk density $M((b_{\eta_i})_{i=1}^n)$ by ρ (see (9))
 $N \in (128, 256, 512, 1024, 2048)$, $L \in (1024, 1024, 1024, 512, 512, 512)$
 $n = 5$ for all $M((b_{\eta_i})_{i=1}^n)$

We can see that the average bulk density grows with ρ up until $\rho = 1/2$:

$$\mathbb{E}(b_\eta) = \begin{cases} \rho & \rho \leq 1/2 \\ 1/2 & \rho > 1/2 \end{cases}$$

The bulk density describes how tight the particles are sitting on the part of the lattice without clusters; if it is $1/4$ for every 4 cluster-less sites on the lattice one has a particle on it. The fact that the bulk density is not growing with ρ for $\rho > 1/2$ means that the remaining particles are positioned in cluster sites.

We can test this by plotting $\tilde{b}_\eta$, the fraction of sites without clusters, for varying ρ like before. We can see the result in figure 5. The apparent relationship between ρ and $\tilde{b}$ is plotted in red. We will later, in section 6, discuss how we arrived at that result, but for now it is more important to see that there only appear to be clusters for $\rho > 1/2$. We will examine this more closely in the next section.

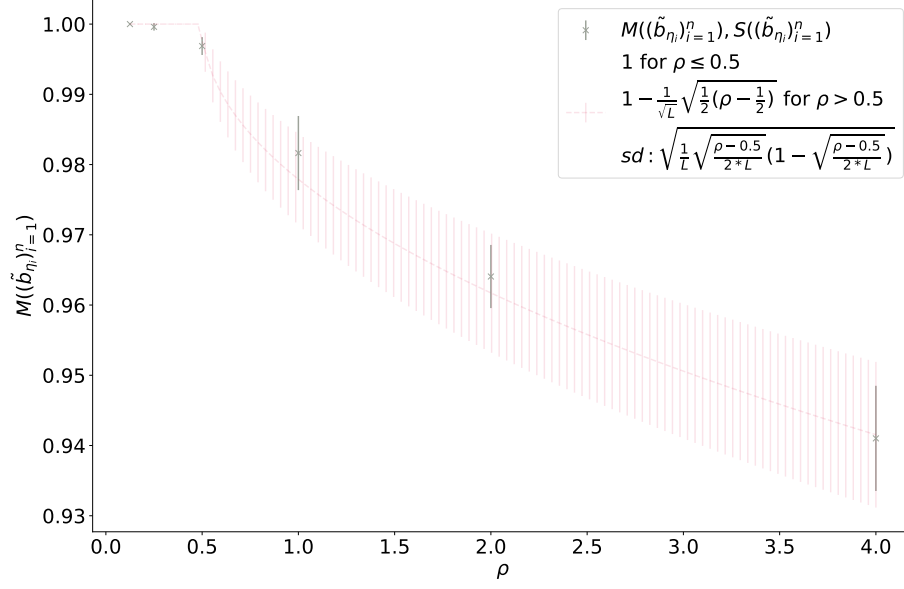


Figure 5: The fraction of sites without clusters (see (10))
 $N \in (128, 256, 512, 1024, 2048), L \in (1024, 1024, 1024, 512, 512, 512)$
 $n = 5$ for all $M((\tilde{b}_{\eta_i})_{i=1}^n)$

5.2 Clusters

5.2.1 Cluster Size

For $\eta \in X_{L,N}$

$$c_\eta = \left\{ \frac{\eta(x)}{\sqrt{L}} : x \in \Lambda_L, \eta(x) > 1 \right\} \quad (11)$$

are called the standardized cluster sizes.

We can first look at $M((c_{\eta_i})_{i=1}^n)$ and $S((c_{\eta_i})_{i=1}^n)$ for a few different values of L for constant $\rho = 1$ (so $N = L$). We will plot this in figure 6. We see that $M((c_{\eta_i})_{i=1}^n) \approx 1$ and the standard deviation does also not depend on L , suggesting that each coordinate of c_η converges to a random variable in the thermodynamic limit, $L, N \rightarrow \infty$ with $N/L \rightarrow \rho$.

For varying ρ , as seen in figure 7, we find that for $\rho \leq 1/2$ the cluster sizes are almost zero and the size rises with ρ on a square root curve; again we will discuss the exact relationship later (see section 5.2.2 for the sample average curve and 5.2.3 for the exact distribution), but so far the prediction that clusters only form for $\rho > 1/2$ has been confirmed.

It is also interesting to note, if we compare figures 6 and 7, that the standard deviation seems to be entirely independent of L and increasing with ρ . This observation will be important later when we are looking at the exact distribution of c_η . It is also important to note that the coordinates of c_η are negatively correlated since if there is already a large cluster on the graph, there would not be many particles left for another cluster. However, cluster sizes should be asymptotically independent for L and N large enough, since for an unlimited amount of particles the described correlation should not be relevant, which is also supported by the equivalence of ensembles result.

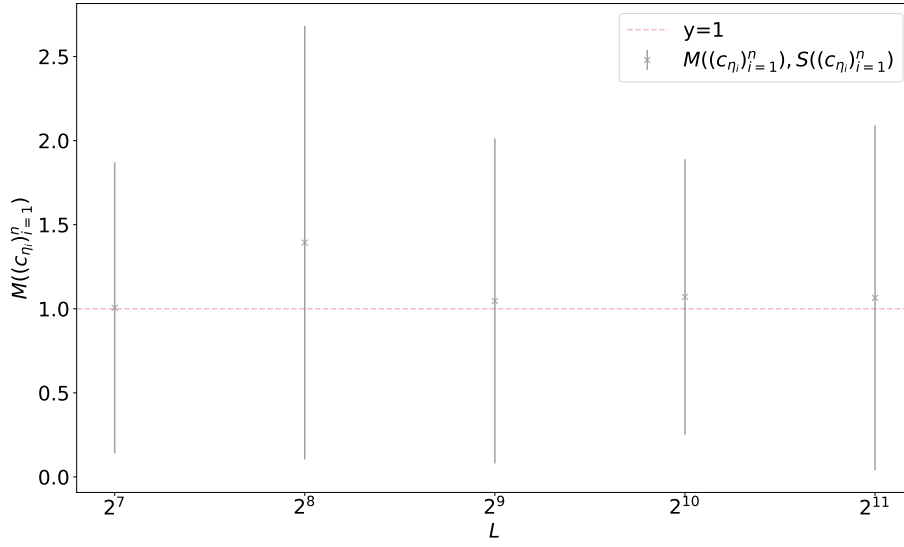


Figure 6: The average standardized cluster size (see (11))
 $L \in (128, 256, 512, 1024, 2048), \rho = 1$
 $n = 6$ for all $M((c_{\eta_i})_{i=1}^n)$

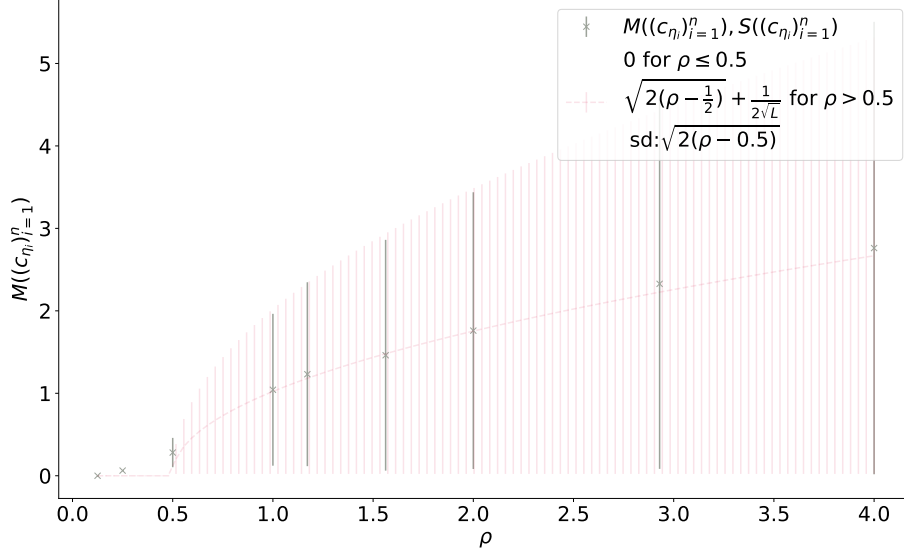


Figure 7: The average standardized cluster size (see (11))
 $N \in (128, 256, 512, 512, 600, 800, 1024, 1500, 2048)$,
 $L \in (1024, 1024, 1024, 512, 512, 512, 512, 512, 512)$
 $n \in (5, 5, 5, 99, 100, 100, 98, 100, 100)$

5.2.2 Cluster Count

For $\eta \in X_{L,N}$

$$a_\eta = \frac{\#\{x : x \in \Lambda_L, \eta(x) > 1\}}{\sqrt{L}} \quad (12)$$

is called the standardized cluster amount.

In figure 8 we see the standardized cluster amount with varying ρ . We see a similar relationship to the standardized cluster size, c_η , and we now have enough data to explain the curve the sample average is on; we will explore the distribution of a_η in more detail in section 6.

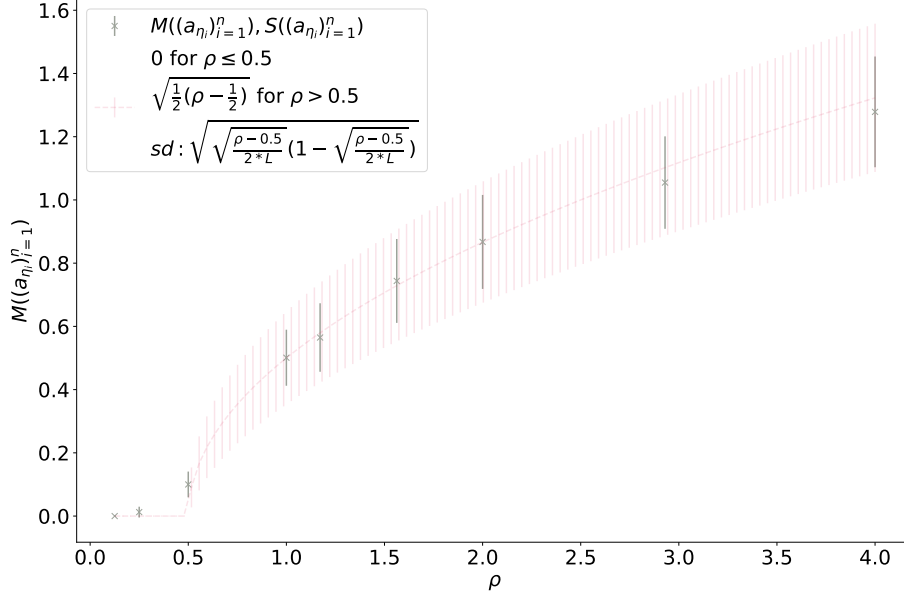


Figure 8: The average standardized cluster amount (see (12))
 $N \in (128, 256, 512, 512, 600, 800, 1024, 1500, 2048)$,
 $L \in (1024, 1024, 1024, 512, 512, 512, 512, 512, 512)$
 $n \in (5, 5, 5, 99, 100, 100, 98, 100, 100)$

By $\mathbb{E}(c_\eta)$, we mean the expected values of any of the entries of the vector c_η , in the following. Before we try to derive any predictions about the observables consider the following representation of the number of total number particles N :

$$\begin{aligned}
N &= \sum_{x \in \Lambda_L} \eta(x) \\
&= \sum_{x \in \Lambda_L} \eta(x) \mathbb{1}_{\{\eta(x) \leq 1\}}(x) + \sum_{x \in \Lambda_L} \eta(x) \mathbb{1}_{\{\eta(x) > 1\}}(x) \\
&= \#\{x \in \Lambda_L : \eta(x) = 1\} + \sum_{i=1}^{a_\eta \sqrt{L}} c_{\eta,i} \sqrt{L} \\
&= L b_\eta \tilde{b}_\eta + \sum_{i=1}^{a_\eta \sqrt{L}} c_{\eta,i} \sqrt{L}
\end{aligned}$$

that means:

$$\begin{aligned}
N = \mathbb{E}(N) &= \mathbb{E}\left(Lb_\eta\tilde{b}_\eta + \sum_{i=1}^{a_\eta\sqrt{L}} c_{\eta,i}\sqrt{L}\right) \\
&= L\mathbb{E}(b_\eta\tilde{b}_\eta) + \mathbb{E}\left(\sum_{i=1}^{a_\eta\sqrt{L}} c_{\eta,i}\sqrt{L}\right) \\
&\stackrel{(*)}{=} L\mathbb{E}(b_\eta)\mathbb{E}(\tilde{b}_\eta) + L\mathbb{E}(a_\eta)\mathbb{E}(c_\eta)
\end{aligned}$$

(*) follows, for the first part of the summation, from the equivalence of ensembles result and the second part follows from Wald's equation.

For $\rho \leq 1/2$

$$\mathbb{E}(b_\eta) = \rho$$

and

$$\mathbb{E}(\tilde{b}_\eta) = 1$$

meaning that

$$\mathbb{E}(\#\{x \in \Lambda_L : \eta(x) = 1\}) = L\mathbb{E}(b_\eta\tilde{b}_\eta) = L\rho = N$$

so all the particles are sitting on individual sites, therefore $\mathbb{E}(c_\eta) = \mathbb{E}(a_\eta) = 0$, as we have seen in graphs 7 and 8.

For $\rho > 1/2$

$$\mathbb{E}(b_\eta) = \frac{1}{2}$$

so

$$\mathbb{E}(\#\{x \in \Lambda_L : \eta(x) = 1\}) = \frac{L}{2}\mathbb{E}(\tilde{b}_\eta)$$

we also know that the expected amount of clusters $\sqrt{L}\mathbb{E}(a_\eta)$ times the expected size of a cluster $\sqrt{L}\mathbb{E}(c_\eta)$ must be the total amount of particles minus the particles in the bulk, as reasoned before:

$$\begin{aligned}
L\mathbb{E}(a_\eta)\mathbb{E}(c_\eta) &= N - \frac{L}{2}\mathbb{E}(\tilde{b}_\eta) \\
\iff \mathbb{E}(a_\eta)\mathbb{E}(c_\eta) &= \rho - \frac{1}{2}\mathbb{E}(\tilde{b}_\eta)
\end{aligned}$$

And we know that the expected amount of non-cluster sites $\mathbb{E}(L\tilde{b}_\eta)$ must be equal to $L - \mathbb{E}(\sqrt{L}a_\eta)$, the total sites minus the sites with clusters.

$$\begin{aligned}
\mathbb{E}(L\tilde{b}_\eta) &= L - \mathbb{E}(\sqrt{L}a_\eta) \\
\iff \mathbb{E}(\tilde{b}_\eta) &= 1 - \frac{1}{\sqrt{L}}\mathbb{E}(a_\eta)
\end{aligned}$$

By combining these, we get

$$\begin{aligned}
\mathbb{E}(a_\eta)\mathbb{E}(c_\eta) &= \rho - \frac{1}{2}\mathbb{E}(\tilde{b}_\eta) \\
\iff \mathbb{E}(a_\eta)\mathbb{E}(c_\eta) &= \rho - \frac{1}{2}\left(1 - \frac{1}{\sqrt{L}}\mathbb{E}(a_\eta)\right) \\
\iff \mathbb{E}(a_\eta)\left(\mathbb{E}(c_\eta) - \frac{1}{2\sqrt{L}}\right) &= \rho - \frac{1}{2}
\end{aligned}$$

because of the root curve shape of the data, and those relationships, we can try the following formulas.

$$\begin{aligned}
\mathbb{E}(a_\eta) &= \frac{1}{D}\sqrt{\rho - \frac{1}{2}} \\
\mathbb{E}(c_\eta) &= D\sqrt{\rho - \frac{1}{2}} + \frac{1}{2\sqrt{L}} \\
\mathbb{E}(\tilde{b}_\eta) &= 1 - \frac{1}{\sqrt{L}}\frac{1}{D}\sqrt{\rho - \frac{1}{2}}
\end{aligned}$$

To solve for D we can use figure 6; for $\rho = 1$, $\mathbb{E}(c_\eta)$ must be about 1, so $D \approx \left(\sqrt{1 - \frac{1}{2}}\right)^{-1} = \sqrt{2}$. Summing up all these results leads to

$$\begin{aligned}
\mathbb{E}(a_\eta) &= \sqrt{\frac{\rho - 1/2}{2}} \\
\mathbb{E}(c_\eta) &= \sqrt{2\rho - 1} + \frac{1}{2\sqrt{L}} \\
\mathbb{E}(\tilde{b}_\eta) &= 1 - \sqrt{\frac{\rho - 1/2}{2L}}
\end{aligned}$$

as has been confirmed in figures 6 to 8.

Now it would be interesting to know the exact distribution of c_η ; we will look into that in the next section.

5.2.3 Cluster Size Tail

Conjecture

The standardized cluster size in the thermodynamic limit, C , is exponentially distributed, so for $\eta \sim \pi_{L,N}$

$$\begin{aligned} c_{\eta,i} &\xrightarrow{d} C \sim \mathcal{E}_{\sqrt{2\rho-1}} \text{ for } i = 1 \dots \#c_\eta, \\ \mathbb{E}(C) &= \sqrt{2\rho-1}, \\ \mathbb{V}(C) &= 2\rho-1. \end{aligned}$$

For $\eta_i \in X_{L,N}, i = 1 \dots n$

$$t_\eta(s) = \frac{\#\{x \in \Lambda_L : \eta(x) > \sqrt{2N - L}s\}}{\#\{x \in \Lambda_L : \eta(x) > 1\}} \quad (13)$$

is called the cluster tail.

The cluster tail is supposed to give us insight into a random variable C that is the limiting distribution of $\pi_{L,N}$ as L approaches ∞ . We know that $\mathbb{E}(c_\eta) \rightarrow \mathbb{E}(C)$, for $L \rightarrow \infty$, and $\mathbb{P}(\sqrt{L}C > \sqrt{2N - L}s) \approx t_\eta(s)$.

We will write $t_{(\eta_i)_{i=1}^n}(s)$ if we combined multiple realizations of η to get a better resolution for our graphs, it describes, analogous to $t_\eta(s)$, the number of sites where any η_i is greater than $\sqrt{2N - L}s$ divided by the total number of cluster sites within all η_i . In figure 9, we see $t_{(\eta_i)_{i=1}^n}(s)$ plotted regularly, and on a logarithmic scale below. We can see the plot follows an exponential curve; the lighter points are the tails for $t_{\eta_i}(s)$ for $i = 1 \dots n$.

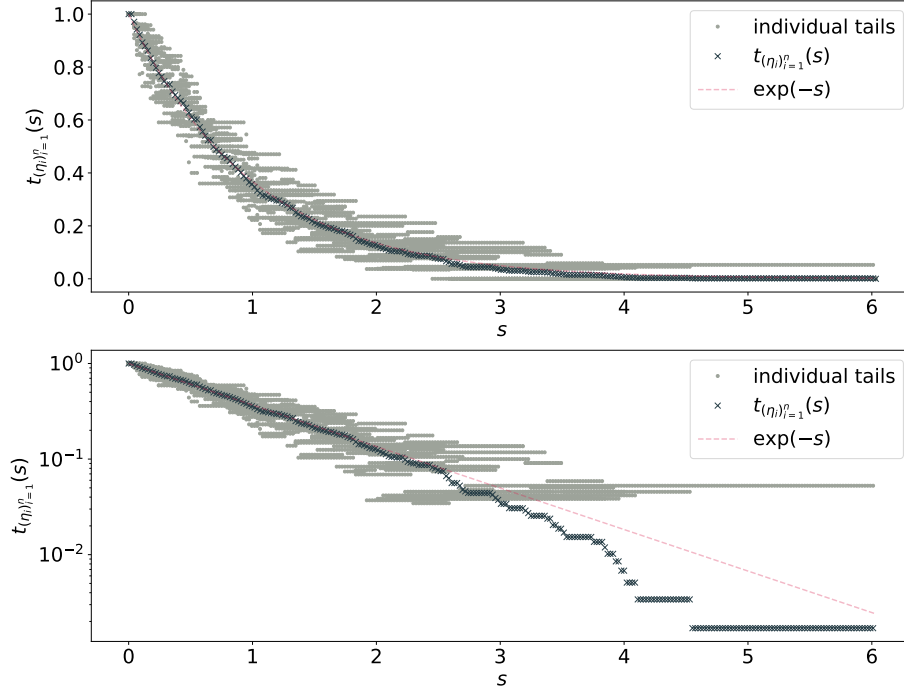


Figure 9: The cluster tail (see (13))
 $L = 2048, \rho = 1, n = 25$

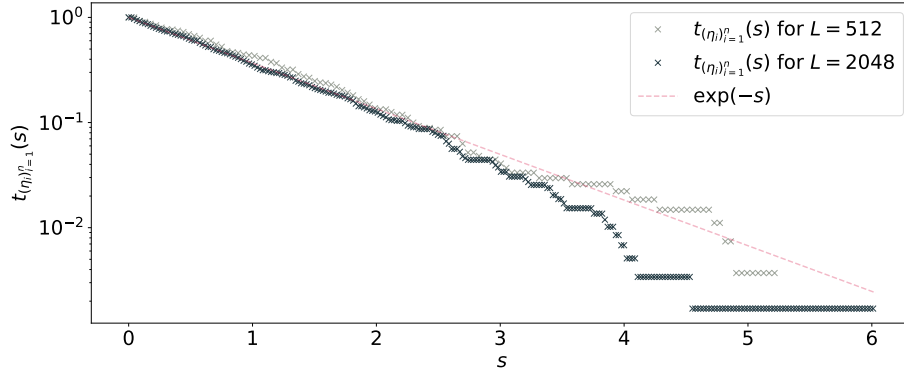


Figure 10: The cluster tail (see (13))
 $L \in (512, 2048), \rho = 1, n = 25$

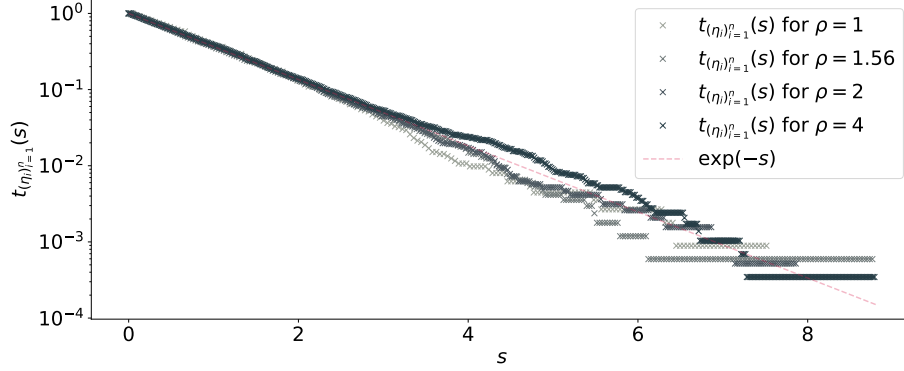


Figure 11: The cluster tail (see (13))
 $N \in (512, 800, 1024, 2048), L \in (512, 512, 512, 512)$
 n rising order of ρ : 99, 100, 98, 100

In figure 10 we plotted two realisations of $t_{(\eta_i)_{i=1}^n}(s)$ with $\rho = 1$, but varying N and L and figure 11 shows $t_{(\eta_i)_{i=1}^n}(s)$ for varying ρ . They also all clearly follow the exponential curve.

So

$$\mathbb{P}(\sqrt{LC} > \sqrt{2N - L}s) = \exp(-s)$$

or

$$\mathbb{P}(C > s) = \exp\left(-\frac{\sqrt{L}s}{\sqrt{2N - L}}\right) = \exp\left(\frac{-s}{\sqrt{2\rho - 1}}\right),$$

which means C is an exponentially distributed random variable with expected value $\sqrt{2\rho - 1}$. This is consistent with the previous result:

$$\mathbb{E}(c_\eta) = \sqrt{2\rho - 1} + \frac{1}{2\sqrt{L}} \rightarrow \sqrt{2\rho - 1} = \mathbb{E}(C) \text{ for } L \rightarrow \infty$$

6 Additional Results for Cluster Count and Bulk

Conjecture

The standardized cluster amount for large L , a_η , is normally distributed:

$$\frac{a_\eta - m}{\sqrt{v}} \xrightarrow{d} \mathcal{N}_{0,1} \text{ where} \quad (14)$$

$$m := \mathbb{E}(a_\eta) = \sqrt{\frac{\rho - \frac{1}{2}}{2}} \quad (15)$$

$$v := \mathbb{V}(a_\eta) = \frac{\mathbb{E}(a_\eta)}{\sqrt{L}} \left(1 - \frac{\mathbb{E}(a_\eta)}{\sqrt{L}}\right) \quad (16)$$

and the fraction of sites without clusters for large L , $\tilde{b}$, is also normally distributed:

$$\frac{\tilde{b}_\eta - m}{\sqrt{v}} \xrightarrow{d} \mathcal{N}_{0,1} \text{ where}$$

$$m := \mathbb{E}(\tilde{b}_\eta) = 1 - \sqrt{\frac{\rho - \frac{1}{2}}{2L}}$$

$$v := \mathbb{V}(\tilde{b}_\eta) = \frac{\mathbb{V}(a_\eta)}{L}$$

Because of the spatial homogeneity of the ZRP, we can assume that any of the L sites have or do not have a cluster independently of each other. Then we can model whether or not a site has a cluster as a Bernoulli distributed random variable X_i ; $X_i = 1$, if the site has a cluster, and $X_i = 0$ otherwise. The sum of these random variables is the number of clusters on the lattice and a binomially distributed random variable $X \sim \mathcal{B}_{L,p_L}$.

$$\begin{aligned} p_L &= \frac{1}{L} \mathbb{E}(X) \\ &= \mathbb{E}(\#\{x \in \Lambda_L : \eta(x) > 1\}) \\ &= \frac{1}{\sqrt{L}} \mathbb{E}\left(\frac{\#\{x \in \Lambda_L : \eta(x) > 1\}}{\sqrt{L}}\right) \\ &= \frac{1}{\sqrt{L}} \mathbb{E}(a_\eta) \\ &= \sqrt{\frac{\rho - 1/2}{2L}} \end{aligned}$$

We can derive the variance of a_η similarly:

$$\begin{aligned}
\mathbb{V}(a_\eta) &= \mathbb{V}\left(\frac{X}{\sqrt{L}}\right) \\
&= \frac{\mathbb{V}(X)}{L} \\
&= \frac{Lp_L(1-p_L)}{L} \\
&= p_L(1-p_L)
\end{aligned}$$

This result works very well with the data in figure 8, as well as the data in figure 4 with the formula for $\mathbb{V}(\tilde{b}_\eta)$ we get in (18). Also notice that $\mathbb{V}(a_\eta) = \mathbb{V}(X_i)$ for any $i = 1 \dots L$, which means

$$\begin{aligned}
\frac{a_\eta - \mathbb{E}(a_\eta)}{\sqrt{\mathbb{V}(a_\eta)}} &= \frac{\frac{X}{\sqrt{L}} - \mathbb{E}\left(\frac{X}{\sqrt{L}}\right)}{\sqrt{\mathbb{V}(X_i)}} \\
&= \frac{X - \mathbb{E}(X)}{\sqrt{L\mathbb{V}(X_1)}} \\
&= \frac{\sum_{i=1}^L X_i - \mathbb{E}\left(\sum_{i=1}^L X_i\right)}{\sqrt{L\mathbb{V}(X_1)}} \\
&= \frac{1}{\sqrt{L}} \sum_{i=1}^L \frac{X_i - \mathbb{E}(X_i)}{\sqrt{\mathbb{V}(X_i)}} \xrightarrow{d} Y \sim \mathcal{N}_{0,1} \text{ for } L \rightarrow \infty
\end{aligned}$$

by the central limit theorem. So for large L we expect a_η to be normally distributed with $\mathbb{E}(a_\eta)$ and $\mathbb{V}(a_\eta)$ as defined in (15) and (16) respectively.

We can validate these results numerically by comparing the empirical density function of a_η to the cumulative density function of a normal distribution as seen in figure 12 or looking at a q-q plot of standardized realizations of a_η and a normal distribution as seen in figure 13. Both plots support the theoretical results; in figure 12 the empirical result matches up closely with the theoretical, and more importantly in the q-q plot in figure 13, we find the empirical quantiles match the theoretical ones of a normal distribution quite well.

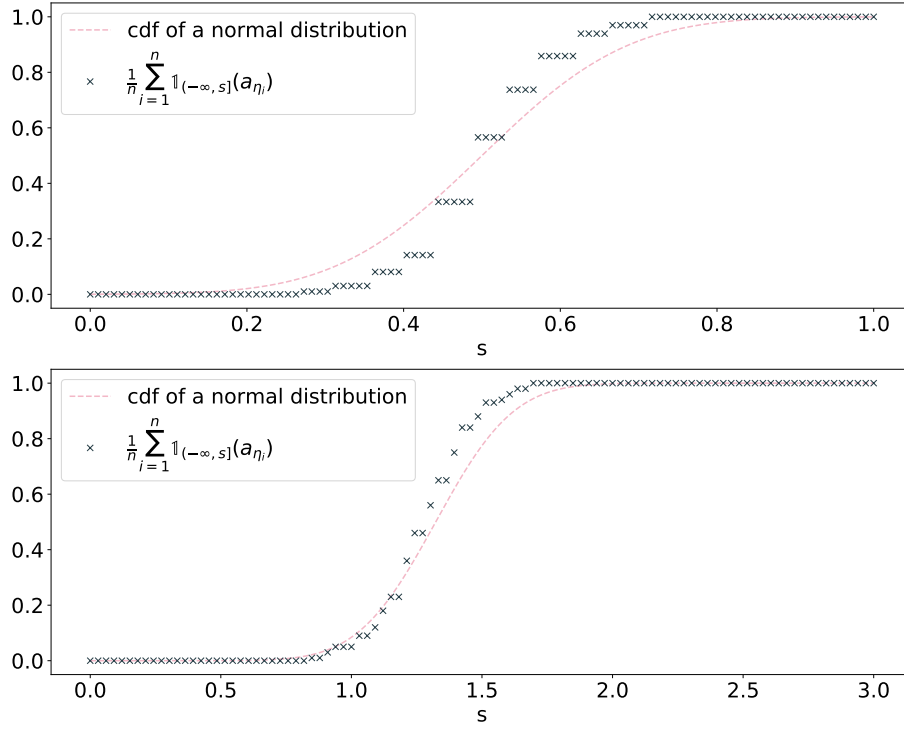


Figure 12: The empirical distribution function of n realisations of a_η ,
normal distribution with $\mathbb{E}(a_\eta)$ and $\mathbb{V}(a_\eta)$ (see (12), (15) and (16))
 $L = 512, N \in (512, 2048), n \in (99, 100)$
 N and n from top to bottom

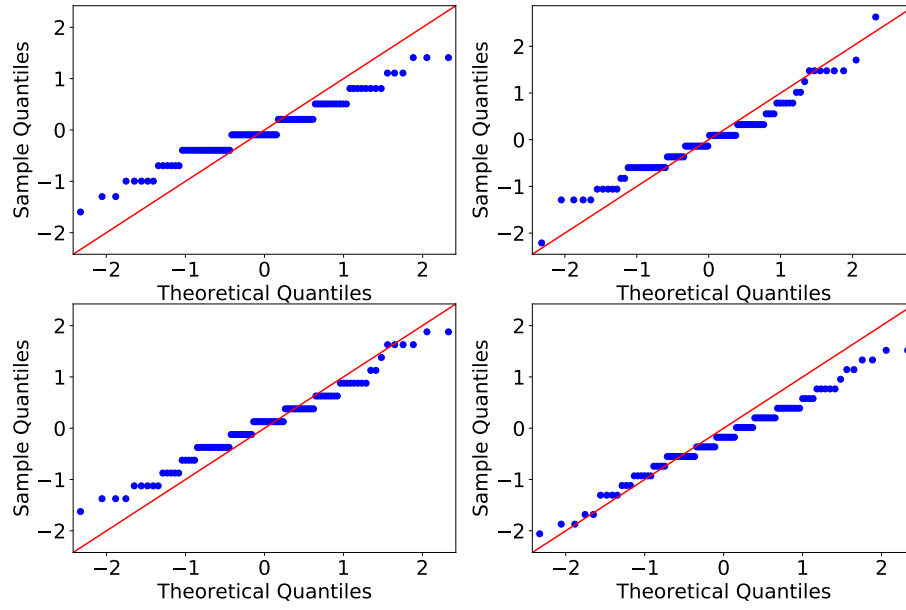


Figure 13: The q-q plots of n realisations of $\frac{a_n - \mathbb{E}(a_n)}{\sqrt{V(a_n)}}$ respectively (see (12))
 $L = 512, N \in (512, 800, 1024, 2048), n \in (99, 100, 98, 100)$
 N and n left to right, top to bottom

For the thermodynamic limit, $L, N \rightarrow \infty$ with $N/L \rightarrow \rho$, we find

$$\begin{aligned} \frac{1}{L} \sum_{i=1}^L X_i &\xrightarrow{d} \mathbb{E}(X_i) \\ \Rightarrow \frac{1}{\sqrt{L}} \sum_{i=1}^L X_i &\xrightarrow{d} \mathbb{E}(a_\eta) \\ \Rightarrow a_\eta &\xrightarrow{d} \mathbb{E}(a_\eta) \end{aligned}$$

since $\mathbb{E}(a_\eta) = \sqrt{L}p_L = \sqrt{L}\mathbb{E}(X_i)$. We are using a weaker version of the law of large numbers and only make a statement about convergence in distribution, even though Skorokhod's representation theorem allows a version with almost sure convergence. It is more sensible here to only consider this version since there is no natural common probability space of the X_i for varying L .

We also know the relationship between $\tilde{b}_\eta$ and a_η

$$\begin{aligned} L &= L\tilde{b}_\eta + \sqrt{L}a_\eta & (17) \\ \Leftrightarrow \tilde{b}_\eta &= 1 - \frac{a_\eta}{\sqrt{L}} & (18) \end{aligned}$$

from which similar results for $\tilde{b}_\eta$ follow trivially.

7 Conclusion

A Zero-range process is a mathematical model used to understand how a group of things, such as particles, move on a graph over time. In a Zero-range process, the rate at which the particles move from one place to another is determined by the number of particles present at the current location.

We study the Zero-range process as a random process of configurations on a lattice and how to understand this process in terms of its generator (see section 4) and examine the unique stationary distribution of the Zero-range process and its behavior on large lattices with respect to the number of particles by lattice size quotient, ρ . We found that the particles distribute significantly differently if ρ exceeds $1/2$, then in the case $\rho \leq 1/2$ where $\pi_{N,L} \approx \nu_\phi$ in the thermodynamic limit, as discussed in section 4 in the theorem *Equivalence of Ensembles in the Thermodynamic Limit* and thereafter and explored those concepts in each case numerically.

We found that clusters only form for $\rho > 1/2$, and examined the distribution of their size c_η (see (11)) and the number a_η (see (12)) in sections 5.2.2, 5.2.3 and 6. We found that the fraction of sites without clusters, b_η (see (10)), tends to one in the thermodynamic limit, meaning the number of sites with clusters

on the lattice goes to zero.

To prove conjectured results on the cluster size distribution etc. involves a detailed analysis of the canonical measure $\pi_{L,N}$. More generally it is interesting to consider, jump rates,

$$g(n) = \begin{cases} 0 & \text{if } n = 0 \\ \theta L^\alpha & \text{if } n = 2 \\ 1 & \text{otherwise} \end{cases}$$

with a constant $\theta > 0$ and power $\alpha > 0$ and see how this affects the typical cluster scale and distribution in analogy to results in [3], where similar generalizations led to different speeds of the dynamics within the clusters and in the bulk than in the specialized case.

8 Appendix: Simulation of the Zero-Range Process in C++

Mathematics and Computer science are closely related fields, which has led some mathematicians to adopt math-like variable naming conventions in their programming, using short and nondescript, mostly one-letter names, for their variables. This is fine for proofs, as they are often self-contained, and the meaning of the variable is accessible within the relative context of the proof but that is not the case for programming. Code readability and maintainability are important in modern programming, and variable and function names should be descriptive and extensive enough to tell the reader what they are used for without extra context. In the following, we will explain which of the variables and functions in the code correspond to which variable and function in the mathematical context where it applies.

- *Configuration* is η
- *LatticeSize* is L
- *ParticleCount* is N
- *Rate*($\cdot$) is $g(\cdot)$

At the beginning of the simulation, we put *ParticleCount*-amount of particles on a field, *Configuration*, of length *LatticeSize*. How exactly we initialize the field is not important, since the stationary distributions are independent of the initial conditions on the lattice, as seen in 14, so we just put all the particles on the first site.

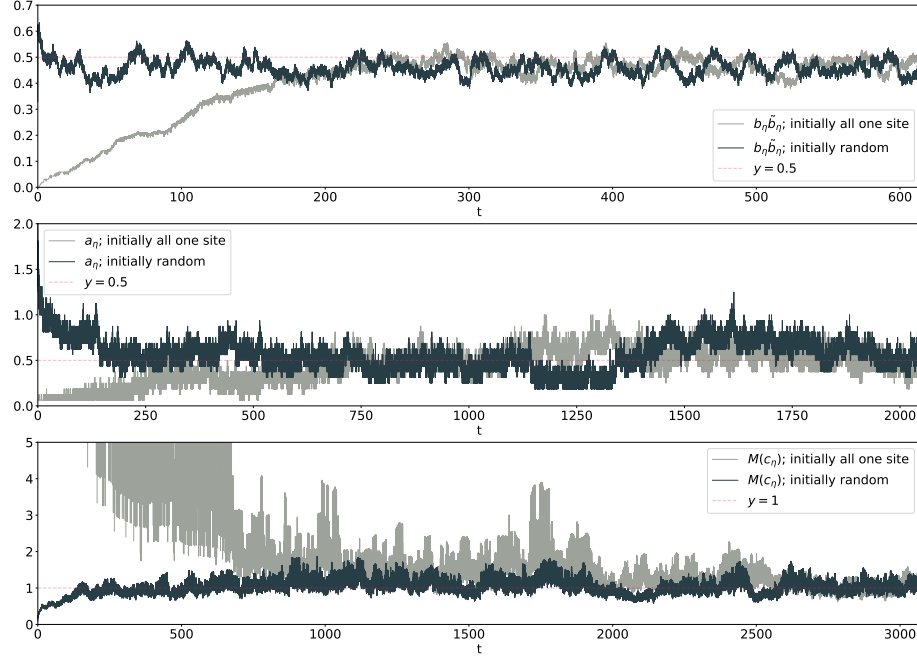


Figure 14: The difference between initial conditions over time
 $\rho = 1, L = 512$

```
Configuration[0] = ParticleCount;
for (int i = 1; i < LatticeSize; i++)
    Configuration[i] = 0;
```

We then loop over an update function, increasing the *CurrentTime* variable in each iteration until the time reaches the *CompletionTime*. The *CompletionTime* has to be large enough, otherwise, the resulting configuration will not be $\pi_{N,L}$ -distributed. In this thesis, we are using $CompletionTime = LatticeSize^2$ for all simulations.

```
CurrentTimeCurrentTime = 0
while(CurrentTime < CompletionTime)
    Update();
```

In the update function, we have to advance the time, choose the site from which the particle leaves and choose the destination site of the particle.

```

void Update()
{
    //add time passed since the last update to CurrentTime
    int rateSum = 0
    for (int i = 1; i < LatticeSize; i++)
        rateSum += Rate(Configuration[i])

    CurrentTime += 1.0 / rateSum;

    //select the site at which a particle is leaving
    int particleOriginSite = Random.GetOrigin();

    //select the distance the particle is traveling
    int particleTravelDistance
        = Random.GetIntegerBelow(LatticeSize) + 1;

    //select the site to which the particle will be going
    int particleDestinationSite
        = particleOriginSite + particleTravelDistance;

    particleDestinationSite %= LatticeSize;

    //update configuration
    Configuration[particleOriginSite]--;
    Configuration[particleDestinationSite]++;
}

```

Random.GetIntegerBelow(C) generates a uniformly distributed integer in $\{0, \dots, C-1\}$. *Random.GetOrigin()* gets a random site from which the particle is leaving; the site is chosen such that any site $x \in \lambda$ is chosen with probability

$$p(x) = \frac{g(\eta(x))}{\sum_{y \in \Lambda} g(\eta(y))}$$

We add $\left(\sum_{x \in \Lambda} g(\eta(x))\right)^{-1}$ to the time in every update instead of realizing an exponential variable with parameter $rateSum = \sum_{x \in \Lambda} g(\eta(x))$. We do this since in this case, the runtime of the program is important since the *LatticeSize* is so large and thus the *CompletionTime* is as well, and for a large amount of exponentially distributed variables, their sum will be close to the sum of their expected values anyway.

$$\begin{aligned}
 &X_i \sim \mathcal{E}_\alpha \text{ i.i.d.} \\
 &\Rightarrow \frac{1}{n} \sum_{i=1}^n X_i \xrightarrow{\text{a.s.}} \frac{1}{\alpha} \text{ for } n \rightarrow \infty
 \end{aligned}$$

We can implement *Random.GetOrigin()* with a very fast runtime if we store all the rates of the current configuration in a tree, such that the leaf nodes are the jump rates at a site and every two nodes have a parent that is the sum of the values of each child, as seen in figure 15.

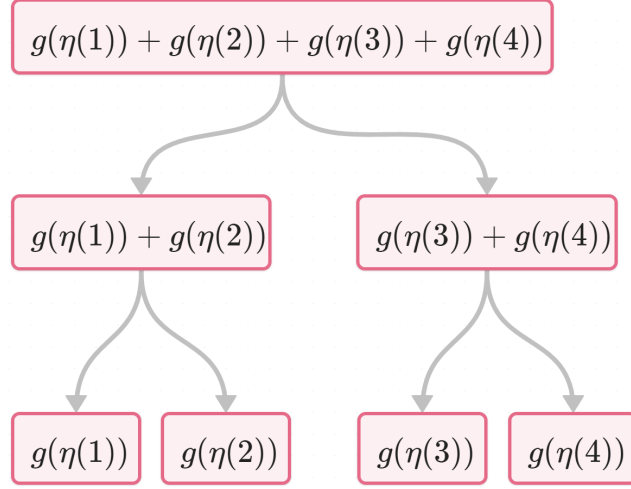


Figure 15: Binary tree, storing the jump rates of the current configuration in the leaf nodes where the parent nodes are the sum of their two child nodes

If we do this, we only have to realize a uniformly distributed random variable on $[0, rateSum]$ and then perform a binary search in the tree, going left if the realization is smaller than or equal to the value in the next left node and right otherwise, if we move right we subtract the value of the left node. If we implement this tree as an array by concatenating each layer of the tree, we move left by multiplying the current list index by two and right if we multiply by two and add one; we have $LatticeSize$ leaf nodes and initialize the tree array, $Tree$, with size $2LatticeSize$. The selected site is the index of the selected leaf node minus the $LatticeSize$ (since the first $LatticeSize$ items of $Tree$ are the parent nodes).

```

int Random::GetOrigin()
{
    double randomNumber = Random.GetDoubleBelow(Tree[1]);
    int i = 1;
    while (i < LatticeSize)
    {
        i <<= 1;
        if (Tree[i] < randomNumber)
        {
            randomNumber -= Tree[i];
            i++;
        }
    }
    return i - LatticeSize;
}

```

If we implement our site selection like so, we also have to update the tree every time a particle moves, which can be done efficiently by remembering at which sites the particle counts changed and only updating the nodes that are affected by this change. We also have to restrict ourselves to lattices of size 2^n , otherwise, the tree format is not possible.

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